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# Bounds on graph eigenvalues I

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## Abstract

We improve some recent results on graph eigenvalues. In particular, we prove that if  $G$  is a graph of order  $n \geq 2$ , maximum degree  $\Delta$ , and girth at least 5, then

$$\mu(G) \leq \min\{\Delta, \sqrt{n-1}\},$$

where  $\mu(G)$  is the largest eigenvalue of the adjacency matrix of  $G$ .

Also, if  $G$  is a graph of order  $n \geq 2$  with dominating number  $\gamma(G) = \gamma$ , then

$$\lambda_2(G) \leq \begin{cases} n & \text{if } \gamma = 1, \\ n - \gamma & \text{if } \gamma \geq 2, \end{cases}$$

$$\lambda_n(G) \geq \lceil n/\gamma \rceil,$$

where  $0 = \lambda_1(G) \leq \lambda_2(G) \leq \dots \leq \lambda_n(G)$  are the eigenvalues of the Laplacian of  $G$ .

We also determine all cases of equality in the above inequalities.

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## 1. Introduction

Our notation is standard (e.g., see [1,3]); in particular, we write  $G(n)$  for a graph of order  $n$  and  $G(n, m)$  for a graph of order  $n$  and  $m$  edges. Given a vertex  $u \in V(G)$ , we write  $\Gamma(u)$  for the

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set of neighbors of  $u$  and set  $d(u) = |\Gamma(u)|$ . If  $X, Y$  are two disjoint subsets of  $V(G)$ , we denote by  $e(X, Y)$  the number of  $X - Y$  edges. Given a graph  $G$  of order  $n$ , we write  $\mu(G) = \mu_1(G) \geq \dots \geq \mu_n(G)$  for the eigenvalues of its adjacency matrix and  $0 = \lambda_1(G) \leq \dots \leq \lambda_n(G) = \lambda(G)$  for the eigenvalues of its Laplacian.

This note is motivated by some recent papers on graph eigenvalues. Lu et al. [11] proved that if  $G = G(n)$  is a connected graph of girth at least 5 and maximum degree  $\Delta$ , then

$$\mu(G) \leq \frac{-1 + \sqrt{4n + 4\Delta - 3}}{2}; \quad (1)$$

equality holds if and only if  $G = C_5$ .

Observe that equality holds in (1) also for  $K_2$  and all  $\Delta$ -regular Moore graphs of diameter 2. Hoffman and Singleton [9] proved that  $r$ -regular Moore graphs of diameter 2 exist for  $r = 2, 3, 7$  and possibly 57.

A stronger theorem follows from a result in [5].

**Theorem 1.** *Let  $G = G(n)$  be a graph of maximum degree  $\Delta$  and girth at least 5. Then*

$$\mu(G) \leq \min \{ \Delta, \sqrt{n-1} \}. \quad (2)$$

*Equality holds if and only if one of the following conditions holds:*

- (i)  $G = K_{1,n-1}$ ;
- (ii)  $G$  is a  $\Delta$ -regular Moore graph of diameter 2;
- (iii)  $G = G_1 \cup G_2$ , where  $G_1$  is  $\Delta$ -regular,  $\Delta(G_2) \leq \Delta$ , and the girth of both  $G_1$  and  $G_2$  is at least 5.

Note that the right-hand side of (2) never exceeds the right-hand side of (1).

Given a graph  $G$ , a set  $X \subset V(G)$  is called *dominating*, if  $\Gamma(u) \cap X \neq \emptyset$  for every  $u \in V(G) \setminus X$ . The number  $\gamma(G) = \min\{|X| : X \text{ is a dominating set}\}$  is called the *dominating number* of  $G$ .

Lu et al. [10] proved that if  $n \geq 2$  and  $G = G(n)$  is a connected graph with  $\gamma(G) = \gamma$ , then

$$\lambda_2(G) \leq n - \gamma + \frac{n - \gamma^2}{n - \gamma}.$$

For  $n \geq \gamma^2$  this bound is implied by the following theorem.

**Theorem 2.** *Let  $n \geq 2$  and  $G = G(n)$  be a graph with  $\gamma(G) = \gamma$ . Then*

$$\lambda_2(G) \leq \begin{cases} n & \text{if } \gamma = 1, \\ n - \gamma & \text{if } \gamma \geq 2. \end{cases} \quad (3)$$

*If  $\gamma = 1$ , equality holds if and only if  $G = K_n$ . If  $\gamma = 2$ , equality holds if and only if  $G$  is the complement of a perfect matching. If  $\gamma > 2$ , (3) is always a strict inequality.*

Another result of Lu et al. [10] states that if  $n \geq 2$  and  $G = G(n)$  is a connected graph with  $\gamma(G) = \gamma$ , then

$$\lambda(G) \geq n/\gamma; \quad (4)$$

equality holds if and only if  $K_{1,n-1} \subset G$ .

Inequality (4) follows immediately from a known result stated in Lemma 4 of the same paper – Mohar [12] proved that for every set  $X \subset V = V(G)$ , the inequality  $\lambda(G)|X||V \setminus X| \geq$

$ne(X, V \setminus X)$  holds. Hence, if  $X$  is a dominating set with  $|X| = \gamma$ , then  $e(X, V \setminus X) \geq |V \setminus X| = n - \gamma$ , and (4) follows.

In fact, a subtler theorem holds.

**Theorem 3.** *Let  $n \geq 2$  and  $G = G(n)$  be a graph with  $\gamma(G) = \gamma > 0$ . Then*

$$\lambda(G) \geq \lceil n/\gamma \rceil. \quad (5)$$

*Equality holds if and only if  $G = G_1 \cup G_2$ , where  $G_1$  and  $G_2$  satisfy the following conditions:*

- (i)  $|G_1| = \lceil n/\gamma \rceil$  and  $\gamma(G_1) = 1$ ;
- (ii)  $\gamma(G_2) = \gamma - 1$  and  $\lambda(G_2) \leq \lceil n/\gamma \rceil$ .

Note that the above results of Liu, Lu, and Tian are stated for connected graphs only, illustrating a tendency in some papers on graph eigenvalues to stipulate connectedness a priori – see, e.g. [4,14–18]. If not truly necessary, such stipulation sends a wrong message. For example, Hong [8] stated his famous inequality

$$\mu(G) \leq \sqrt{2e(G) - v(G) + 1}$$

for connected graphs, although his proof works for graphs with minimum degree at least 1. This result has been reproduced verbatim countless times challenging the readers to complete the picture on their own. Confinement to connected graphs simplifies the study of cases of equality, but important points might be missed. As an illustration, recall the result of Hong et al. [7]: if  $G = G(n, m)$  is a connected graph with  $\delta(G) = \delta$ , then

$$\mu(G) \leq \frac{\delta - 1 + \sqrt{8m - 4\delta n + (\delta + 1)^2}}{2}, \quad (6)$$

with equality holding if and only if every vertex of  $G$  has degree  $\delta$  or  $n - 1$ .

Inequality (6) has been proved independently by Nikiforov [13] for disconnected graphs as well; however, as shown in [13,19], there are nonobvious disconnected graphs for which equality holds in (6).

Finally, observe that (6) implies a result of Cao [2], recently reproved for connected graphs by Das and Kumar [4]: if  $G = G(n, m)$  is a graph with  $\delta(G) = \delta \geq 1$  and  $\Delta(G) = \Delta$ , then

$$\mu(G) \leq \sqrt{2m - (n - 1)\delta + (\delta - 1)\Delta}. \quad (7)$$

In fact, (7) follows from (6) by

$$\mu^2(G) \leq 2m - (n - 1)\delta + (\delta - 1)\mu(G) \leq 2m - (n - 1)\delta + (\delta - 1)\Delta.$$

## 2. Proofs

We shall need the following result of Grone and Merris [6].

**Lemma 4.** *If  $G$  is a graph with  $e(G) > 0$ , then  $\lambda(G) \geq \Delta(G) + 1$ ; if  $G$  is connected, then equality holds if and only if  $\Delta(G) = |G| - 1$ .*

**Proof of Theorem 1.** Since  $\mu(G) \leq \Delta(G)$ , to prove (2), all we need is to show that  $\mu(G) \leq \sqrt{n - 1}$ . We follow here the argument of Favaron et al. [5, p. 203]. For every  $u \in V(G)$  set

$$w(u) = \sum_{v \in \Gamma(u)} d(v).$$

As shown in [5, p. 203],  $\mu^2(G) \leq \max_{u \in V(G)} w(u)$ ; if  $G$  is connected, equality holds if and only if  $G$  is regular or bipartite semiregular graph.

We shall prove that  $w(u) \leq n - 1$  for every  $u \in V(G)$ . Indeed, let  $u \in V(G)$ ; for every two distinct vertices  $v, w \in \Gamma(u)$ , in view of  $C_3 \not\subseteq G$  and  $C_4 \not\subseteq G$ , we see that  $e(\Gamma(u)) = 0$  and  $\Gamma(v) \cap \Gamma(w) = \{u\}$ . Hence,

$$w(u) = \sum_{v \in \Gamma(u)} d(v) = e(\Gamma(u), V(G) \setminus \Gamma(u)) \leq d(u) + n - d(u) - 1 = n - 1, \quad (8)$$

completing the proof of (2). Note that if equality holds in (8), then  $\cup_{v \in \Gamma(u)} \Gamma(v) = V(G)$ ; hence, the distance of any vertex  $v \in V(G)$  to  $u$  is at most 2.

Let us determine when equality holds in (2). If any of the conditions (i)–(iii) holds, clearly (2) is an equality. Suppose equality holds in (2). If  $\mu(G) = \Delta$ , then  $G$  contains a  $\Delta$ -regular component, say  $G_1$ . Writing  $G_2$  for the union of the remaining components of  $G$ , we see that (iii) holds, completing the proof in this case.

Now let  $\mu(G) = \sqrt{n} - 1$ ; hence, equality holds in (8) for some vertex  $u \in V(G)$ , implying that  $G$  is connected. According to the aforementioned result of Favaron et al. equality in (8) holds for every vertex  $u \in V(G)$ , and  $G$  is either  $\sqrt{n} - 1$ -regular or bipartite semiregular. Clearly,  $\text{diam } G = 2$ , so if  $G$  is  $\sqrt{n} - 1$ -regular, then it is a Moore graph and (ii) holds.

Finally, let  $G$  be a bipartite graph. Then the distance between any two vertices belonging to different parts of  $G$  is odd; since  $\text{diam } G = 2$ , it follows that  $G$  is a complete bipartite graph, and so,  $G = K_{1, n-1}$ , completing the proof.  $\square$

**Proof of Theorem 2.** Let  $V = V(G)$  and  $X$  be a dominating set with  $|X| = \gamma$ . For the sake of completeness we shall reprove the known inequality  $\delta(G) \leq n - \gamma$ . Indeed, select  $u \in X$ . If  $\Gamma(u) \cap X = \emptyset$ , then  $\Gamma(u) \subset V \setminus X$ , and so  $\delta(G) \leq d(u) \leq n - \gamma$ . Now assume that  $\Gamma(u) \cap X \neq \emptyset$ . Then there exists  $v \in (V \setminus X) \cap \Gamma(u)$  such that  $v$  is not joined to any  $w \in X \setminus \{u\}$ , otherwise  $X \setminus \{u\}$  would be dominating, contradicting that  $X$  is minimal. Hence,

$$\delta(G) \leq d(v) \leq (n - 1) - |X \setminus \{u\}| = n - \gamma$$

as claimed.

If  $G = K_n$ , we have  $\gamma = 1$  and  $\lambda_2(K_n) = n$ , completing the proof. Assume that  $G \neq K_n$ ; hence,  $e(\bar{G}) \geq 1$ . Applying Lemma 4, we have

$$\lambda_2(G) = n - \lambda(\bar{G}) \leq n - \Delta(\bar{G}) - 1 = n - (n - 1 - \delta(G)) - 1 \leq n - \gamma, \quad (9)$$

proving (3).

Let us determine when equality holds in (3). If  $\gamma = 1$  and  $G = K_n$ , then  $\lambda_2(G) = n$ , so (3) is an equality. If  $\gamma = 2$  and  $G = (n/2)K_2$  then  $\lambda_2(G) = n - \lambda(\bar{G}) = n - 2$ , so (3) is an equality.

Suppose now that equality holds in (3). If  $\gamma = 1$ , from  $\lambda_2(G) = n - \lambda(\bar{G}) = n$  we find that  $e(\bar{G}) = 0$ , and so  $G = K_n$ . If  $\gamma \geq 2$ , then we have equalities in (9), implying that  $\delta(G) = n - \gamma$  and  $\lambda(\bar{G}) = \Delta(\bar{G}) + 1 = \gamma$ . From Lemma 4 we conclude that  $\bar{G}$  has a component  $G_1$  such that  $\Delta(G_1) = \gamma - 1$  and  $|G_1| = \gamma$ . Set  $V_1 = V(G_1)$ . Since  $G_1$  is a component of  $\bar{G}$ , the pair  $(V_1, V \setminus V_1)$  induces a complete bipartite graph in  $G$  and so  $\gamma = 2$ . We have  $\lambda(\bar{G}) = n - \lambda_2(G) = 2$  and so  $\Delta(\bar{G}) = 1$ . This implies that  $\bar{G}$  is a perfect matching, as otherwise  $G$  would have a dominating vertex, contradicting that  $\gamma = 2$ . This completes the proof.  $\square$

**Proof of Theorem 3.** Let  $G$  be a graph with  $\gamma(G) = \gamma$ ; set  $V = V(G)$  and let  $G_0 \subset G$  be an edge-minimal subgraph of  $G$  with  $V(G_0) = V$  and  $\gamma(G_0) = \gamma$ . Clearly,  $G_0$  is a union of  $\gamma$  vertex-disjoint stars, and so  $G_0$  contains a star of order at least  $\lceil n/\gamma \rceil$ . Therefore,  $\lambda(G) \geq \lambda(G_0) \geq \lceil n/\gamma \rceil$ , proving (5).

Let us determine when equality holds in (5). If  $G = G_1 \cup G_2$ , where  $G_1$  and  $G_2$  satisfy conditions (i) and (ii) of Theorem 3, then clearly equality holds in (5). Let  $G$  be a graph such that equality holds in (5), and  $G_0 \subset G$  be an edge-minimal subgraph with  $V(G_0) = V$  and  $\gamma(G) = \gamma$ ; clearly,  $G_0$  is a union of  $\gamma$  vertex-disjoint stars, whose centers form a dominating set of  $G$ . From

$$\lceil n/\gamma \rceil = \lambda(G) \geq \lambda(G_0) \geq \lceil n/\gamma \rceil$$

we conclude that  $G_0$  contains a component  $H$  that is star  $K_{1, \lceil n/\gamma \rceil - 1}$ . To complete the proof, we have to show that no edge of  $G$  joins  $H$  to another component  $F$  of  $G_0$ . If there is such an edge, according to Lemma 4, the component  $G'$  of  $G$  containing both  $H$  and  $F$  must satisfy  $\lambda(G') > \lceil n/\gamma \rceil$ , a contradiction. Hence,  $H$  induces a component of  $G$ , say  $G_1$ . We have  $|G_1| = \gamma + 1$  and  $\gamma(G_1) = 1$ , so (i) holds. Setting  $G_2$  for the union of the remaining components of  $G$ , we see that  $\gamma(G_2) = \gamma - 1$ , since  $G_2$  is spanned by  $\gamma - 1$  stars. Observing that  $\lambda(G_2) \leq \lambda(G) = \lceil n/\gamma \rceil$ , condition (ii) follows, completing the proof.  $\square$

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